

# SYMBOLIC EXTENSIONS AND DOMINATED SPLITTINGS FOR GENERIC $C^1$ -DIFFEOMORPHISMS

A. ARBIETO, A. ARMIJO, T. CATALAN, AND L. SENOS

**ABSTRACT.** Let  $\text{Diff}^1(M)$  be the set of all  $C^1$ -diffeomorphisms  $f : M \rightarrow M$ , where  $M$  is a compact boundaryless  $d$ -dimensional manifold,  $d \geq 2$ . We prove that there is a residual subset  $\mathfrak{R}$  of  $\text{Diff}^1(M)$  such that if  $f \in \mathfrak{R}$  and if  $H(p)$  is the homoclinic class associated with a hyperbolic periodic point  $p$ , then either  $H(p)$  admits a dominated splitting of the form  $E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$ , where  $F_i$  is not hyperbolic and one-dimensional, or  $f|_{H(p)}$  has no symbolic extensions.

## 1. INTRODUCTION

*Expansiveness* is an important notion in the theory of dynamical systems. Let  $M$  be a compact manifold and  $f : M \rightarrow M$  be a homeomorphism. Roughly, it says that if the orbits of different points must separate in finite time. More precisely, there exists  $\varepsilon > 0$  such that for any point  $x$ , the  $\varepsilon$ -set of  $x$ , given by the points  $y$  such that  $d(f^n(x), f^n(y)) < \varepsilon$  for every integer  $n$ , reduces to the point  $x$ . This notion is somewhat related with the well know notion of sensitivity to initial conditions, commonly known as *chaos*, which means that for any point, there exists a point such that the future orbit of these two points separated. Moreover, expansiveness naturally appears in hyperbolic sets, and together with the shadowing property, play a central role to prove their stability.

However, it is important to look for weaker forms of expansivity. Clearly, expansivity implies *h-expansivity*, i.e. for some  $\varepsilon > 0$  the entropy of the  $\varepsilon$ -set of any point  $x$  is zero. This notion implies semicontinuity of the entropy map, henceforth leading to the existence of equilibrium states, which is a well know problem in ergodic theory. We remark that *h-expansiveness* do not imply expansivity. This weaker property holds for partially hyperbolic diffeomorphisms such that their central subbundle admits a dominated splitting by one-dimensional subbundles, see [DFPV]. It also holds for diffeomorphisms away from tangencies see [LVY].

It turns out that *h-expansiveness* implies the existence of *symbolic extensions*, see [BFF]. This means that the system is a quotient of a subshift of finite type. Actually, we can ask if the residual entropy of this extension is zero, in this case we say that the extension is *principal*. However, the existence of symbolic extensions does not imply any kind of expansiveness, even asymptotic *h-expansiveness*, which requires that the entropy of the  $\varepsilon$ -sets goes to zero if  $\varepsilon$  goes to zero. In particular, *the non existence* of symbolic extensions implies that a positive amount of entropy, far from zero, can be found in arbitrarily small sets, given some complexity of the dynamics, see [BD]. In the other hand, symbolic extensions are used in the theory

---

Partially supported by CAPES, CNPq, FAPERJ (“Jovem Cientista do Nosso Estado”) and PRONEX/DS from Brazil.

of data transmission, see [D]. It is worthing to remark that any  $C^\infty$  diffeomorphism is asymptotically  $h$ -expansive, see [Buz].

The existence of symbolic extensions is somewhat rare in non-hyperbolic dynamics. Indeed, it was proved by [DN] that  $C^1$ -generic non-Anosov symplectic diffeomorphisms in surfaces do not have symbolic extensions. This result was extended to higher dimensions by [CT]. By generic, we mean that this holds for systems in a residual subset of such diffeomorphisms.

A natural question in dynamical systems is to know whether the presence of a dynamical property in a  $C^1$ -robust way implies some hyperbolicity. For instance, [BDP] shows that robust transitivity implies the existence of a dominated splitting. Naturally, some authors asked this question using expansivity. Indeed, Mañé [M1] shows that any robustly expansive diffeomorphism is Axiom A. The same question can be asked in a semi-local way. More precisely, we can ask if a homoclinic class has some expansiveness in a robust way then it is hyperbolic. By homoclinic class we mean the closure of the transversal homoclinic intersections of a periodic orbit. The series of papers [PPV], [PPSV], [SV1] and [SV2], essentially proves that robustly expansive homoclinic classes are hyperbolic, see the articles for more details. In [PV], it was proved that any robustly  $h$ -expansive homoclinic class has a dominated splitting of the form  $E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$ , where  $F_i$  is not hyperbolic and one-dimensional. A related result was proved by Li in the context of  $R$ -robustly  $h$ -expansive homoclinic classes, see [L] for more details.

Another related question is the existence of a residual subset where the presence of a dynamical property implies hyperbolicity in the global and semi-local case. For instance, in [Ar] it is proved that any generic expansive diffeomorphism is Axiom A. In [C], it was proved that generic volume preserving diffeomorphisms have symbolic extensions if, and only if, they are partially hyperbolic. In the semi-local case, [GY] proved that for a generic diffeomorphisms, any expansive homoclinic class is hyperbolic.

In this article we study these questions for generic diffeomorphisms in the semi-local case but using symbolic extensions, that as we saw before, is much weaker than expansiveness. Another results dealing with the non-existence of symbolic extensions are: [DF] constructed a locally residual subset of  $C^1$ -partially hyperbolic diffeomorphisms without symbolic extensions, [A] also constructed other examples, for smoother systems [DN] conjectured that  $C^r$ -diffeomorphisms have symbolic extensions if  $r \geq 2$ , [Bur2] proved this conjecture for surfaces diffeomorphisms, [BF] extended this result for higher dimensions with 2-dimensional center subbundle. Any  $C^r$ -one-dimensional transformation, with  $r > 1$ , has symbolic extensions, this was proved by [DM].

Now, we give precise definitions and state our main results.

We consider a compact boundaryless  $d$ -dimensional Riemannian manifold  $M$ ,  $d \geq 2$ , and denoting by  $\text{Diff}^r(M)$  the set of  $C^r$  diffeomorphisms on  $M$  endowed with the  $C^r$  topology.

**Definition 1.** A dynamical system  $f : M \rightarrow M$  has a *symbolic extension* if there exists a subshift  $\sigma : N \rightarrow N$  and a continuous surjective map  $\pi : N \rightarrow M$  such that  $\pi \circ \sigma = f \circ \pi$ . In this case the system  $\sigma : N \rightarrow N$  is called an *extension* of  $f : M \rightarrow M$  and  $f$  is called a *factor* of  $\sigma$ . If  $h_{\pi^*\mu}(f) = h_\mu(\sigma)$  for every invariant measure  $\mu$  of  $\sigma$  then the extension is called *principal*.

We say that  $f : M^n \rightarrow M^n$  has a good decomposition in  $\Lambda$  if there exists a dominated splitting  $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$  such that  $\dim(E_1) = s$ ,  $\dim(E_k) = n - u$  and for every  $1 < j < k$  we have  $\dim(E_j) = 1$ . Here,  $s$  (resp.  $u$ ) denotes the smallest (resp. greatest) index of a hyperbolic periodic point in  $\Lambda$ . Recall that the *index* of a hyperbolic periodic point  $p$  is the dimension of its stable manifold.

**Theorem 2.** *There is a residual subset  $\mathcal{R}$  of  $\text{Diff}^1(M)$  such that if  $f \in \mathcal{R}$ , then for every homoclinic class  $H(p, f)$ ,*

- a) *either  $H(p, f)$  has a good decomposition,*
- b) *or  $f|_{H(p, f)}$  has no symbolic extensions.*

To prove this theorem we will use criterions to the non existence of symbolic extensions developed by Downarowicz and Newhouse in [DN]. Moreover, we also use a dichotomy between good decompositions and the existence of a homoclinic tangency, see [ABCDW].

Even so, once that one obtain a good decomposition, is somewhat folklore to obtain partial hyperbolicity when the class is isolated. In particular, we obtain the following theorem and prove it just for sake of completeness.

**Theorem 3.** *There is a residual subset  $\mathcal{R}$  of  $\text{Diff}^1(M)$  such that if  $f \in \mathcal{R}$ , then for every isolated homoclinic class  $H(p, f)$*

- a) *either  $H(p, f)$  is partially hyperbolic,*
- b) *or  $f|_{H(p, f)}$  has no symbolic extensions.*

However, this theorem together with the result of Diaz, Fisher, Pacífico and Vieitez [DFPV] has an interesting directly consequence.

**Corollary 4.** *There is a residual subset  $\mathcal{R}$  of  $\text{Diff}^1(M)$  such that if  $f \in \mathcal{R}$ , any isolated homoclinic class of  $f$  has a symbolic extension if, and only if, it has a principal symbolic extension.*

Finally, as a byproduct of the techniques used in the proof of the main theorem we also get the following interesting consequence, which is somewhat related to the previous result by Pacífico and Vieitez [PV] mentioned before.

**Proposition 5.** *Let  $HT \subset \text{Diff}^1(M)$  be the set of diffeomorphisms exhibiting a homoclinic tangency, and  $NAHE \subset \text{Diff}^1(M)$  be the set of diffeomorphisms that are not asymptotically  $h$ -expansive. Then  $\overline{HT} = \overline{NAHE}$ .*

As a consequence, if a diffeomorphism is stably asymptotically  $h$ -expansive then it has a dominated splitting in the pre-periodic set, using a result of Wen, see [W]. Moreover, if the diffeomorphism is generic then it is partially hyperbolic due to [CSY].

This article is organized as follows: In Section 2, we define precisely the notions and objects used in this paper, in Section 3 we define and study the  $\mathcal{S}_{n,p}$  property, which is our tool to find diffeomorphisms that has no symbolic extensions, in Section 4 we prove a local version of the Theorem 2, in Section 5 we give a proof for Theorem 2, in Section 6 we consider the isolated case and, finally, in Section 7 we prove Proposition 5.

## 2. DEFINITIONS

In this section we define precisely the notions and objects used in the introduction.

We say that  $p$  is a *periodic point* if  $f^n(p) = p$  for some  $n \geq 1$ , the minimal such natural is called the *period* of  $p$  and it is denoted by  $\tau(p, f)$ , or simply by  $\tau(p)$  if the diffeomorphism  $f$  is fixed. The periodic point is *hyperbolic* if the eigenvalues of  $Df^{\tau(p)}(p)$  do not belong to  $S^1$ .

If  $p$  is a hyperbolic periodic point then its *homoclinic class*  $H(p, f)$  is the closure of the transversal intersections of the stable manifold and unstable manifold of the orbit of  $p$ :

$$H(p, f) = \overline{W^s(p) \cap W^u(p)}.$$

It is well known that a homoclinic class is transitive. Moreover, we say that a hyperbolic periodic point  $q$  is *related* to  $p$  if  $W^s(p) \cap W^u(q) \neq \emptyset$  and  $W^u(p) \cap W^s(q) \neq \emptyset$ , it can be proved that the homoclinic class of  $p$  is also the closure of the hyperbolic periodic points related to  $p$ .

Let  $Per_h^n(f)$  be the collection of hyperbolic periodic points of  $f$  of period less than or equal to  $n$ , and let  $Per_h(f) = \bigcup_{n \geq 1} Per_h^n(f)$ .

**2.1. Domination.** We say that a compact  $f$ -invariant set  $\Lambda \subset M$  admits a *dominated splitting* if the tangent bundle  $T_\Lambda M$  has a continuous  $Df$ -invariant splitting  $E_1 \oplus \dots \oplus E_k$  and there exist constants  $C > 0$ ,  $0 < \lambda < 1$ , such that

$$\|Df^n|E_i(x)\| \cdot \|Df^{-n}|E_j(f^n(x))\| \leq C\lambda^n, \quad \forall x \in \Lambda, n \geq 0, \text{ for every } i < j.$$

We say that  $T_\Lambda M = E_1 \oplus \dots \oplus E_k$  is the finest dominated splitting if there is no dominated splitting of  $E_l$  for every  $1 < l < k$ .

**2.2. Hyperbolicity.** If  $\Lambda$  is a compact invariant set of a diffeomorphism  $f$  then  $\Lambda$  is said to be a *hyperbolic set* if we have a  $Df$ -invariant continuous splitting  $T_\Lambda M = E^s \oplus E^u$  and constants  $C > 0$  and  $\kappa < 1$  such that

$$\|Df^{-n}(x)|_{E_x^u}\| \leq C\kappa^n \text{ and } \|Df^n(x)|_{E_x^s}\| \leq C\kappa^n,$$

for every  $x \in \Lambda$  and  $n \in \mathbb{N}$ .

Let  $E \oplus F_1 \oplus \dots \oplus G$  be a dominated splitting over  $\Lambda$ . If  $E$  contracts and  $G$  expands, like in the previous paragraph then we say that  $\Lambda$  is *partially hyperbolic*.

Let  $\Lambda$  be a hyperbolic set for  $f$ . We call  $\Lambda$  a *hyperbolic basic set* if

- it is *isolated*, i.e. there is a neighborhood  $U$  of  $\Lambda$  such that

$$\bigcap_{n \in \mathbb{Z}} f^n(U) = \Lambda \text{ and}$$

- $f$  has a dense orbit in  $\Lambda$ .

**2.3. Genericity.** We say that a subset  $\mathfrak{R} \subset \text{Diff}^1(M)$  is a *residual subset* if contains a countable intersection of open and dense sets.

The countable intersection of residual subsets is also a residual subset. Since  $\text{Diff}^1(M)$  is a Baire space when endowed with the  $C^1$ -topology, any residual subset of  $\text{Diff}^1(M)$  is dense.

We will say that a property (P) holds *generically* if there exists a residual subset  $\mathfrak{R}$  such that any  $f \in \mathfrak{R}$  has the property (P).

**2.4. Measures and Exponents.** A measure  $\mu$  is  $f$ -invariant if  $\mu(f^{-1}(B)) = \mu(B)$  for every measurable set  $B$ . An invariant measure is ergodic if the measure of any invariant set is zero or one. Let  $\mathcal{M}(f)$  be the space of  $f$ -invariant *probability measures* on  $M$ , and let  $\mathcal{M}_e(f)$  denote the ergodic elements of  $\mathcal{M}(f)$ .

For a hyperbolic periodic point  $p$  of  $f$  with period  $\tau(p)$ , we let  $\mu_p$  denote the periodic measure given by

$$\mu_p = \frac{1}{\tau(p)} \sum_{x \in O(p)} \delta_x$$

where  $O(p)$  denotes the orbit of  $p$  and  $\delta_x$  is the Dirac measure at  $x$ .

A measure  $\mu \in \mathcal{M}(f)$  is called a *hyperbolic measure* for  $f$  if its topological support  $\text{supp}(\mu)$  is contained in a hyperbolic basic set for  $f$ .<sup>1</sup>

Let  $C(M, \mathbb{R})$  be the set of all continuous functions  $h : M \rightarrow \mathbb{R}$ . If  $h \in C(M, \mathbb{R})$  then  $\mu(h) = \int_M h d\mu$ . Let us denote by  $\rho$  the metric on  $\mathcal{M}(f)$  which defines the weak-\* topology as follows. Let  $\phi_1, \phi_2, \dots$  be a countable dense subset of the unit ball in  $C(M, \mathbb{R})$  and set

$$\rho(\mu, \nu) = \sum_{i \geq 1} \frac{1}{2^i} |\mu(\phi_i) - \nu(\phi_i)|.$$

Given a periodic ergodic measure  $\mu_p \in \mathcal{M}_e(f)$ , we denote by  $\chi^+(p, f)$  and  $\chi^-(p, f)$  the smallest positive Lyapunov exponent and the biggest negative Lyapunov exponent of  $\mu_p$ , respectively. Then we define  $\chi(p, f) = \min\{\chi^+(p, f), -\chi^-(p, f)\}$ .

### 3. THE PROPERTY $\mathcal{S}_{n,p}$

In this section, we define and study the  $\mathcal{S}_{n,p}$  property. This property is in the spirit of [DN], in order to find diffeomorphisms that has no symbolic extensions.

**Definition 6.** Given a positive integer  $n$ , we say that a diffeomorphism  $f$  satisfies property  $\mathcal{S}_{n,p}$  if  $p$  is a hyperbolic periodic point of  $f$ , and for any  $\tilde{p} \in \text{Per}_h^n(f)$  related to  $p$  there is a zero dimensional periodic hyperbolic basic set  $\Lambda(\tilde{p}, n) \subset H(p, f)$  for  $f$  with the same index that  $p$ , such that the following happens:

a) there is  $\nu \in \mathcal{M}_e(\Lambda(\tilde{p}, n))$  such that

$$h_\nu(f) > \chi(\tilde{p}, f) - \frac{1}{n},$$

b) for every  $\mu \in \mathcal{M}_e(\Lambda(\tilde{p}, n))$ , we have

$$\rho(\mu, \mu_{\tilde{p}}) < \frac{1}{n}.$$

c) for every hyperbolic periodic point  $q \in \Lambda(\tilde{p}, n)$ , we have

$$\chi(q, f) > \chi(\tilde{p}, f) - \frac{1}{n}.$$

The following result concern about the abundance of diffeomorphisms satisfying property  $\mathcal{S}_{n,p}$  near diffeomorphism with homoclinic classes admitting no dominated splittings.

---

<sup>1</sup>This differs from usual definition, where one ask that the Lyapunov exponents of the measure are non-zero.

**Proposition 7.** *Let  $f$  be a Kupka-Smale generic diffeomorphism with a hyperbolic periodic point  $p$  of index  $i$ . If  $H(p, f)$  is a non-trivial homoclinic class admitting no  $i$ -dominated splitting, then for any neighborhood  $\mathcal{U}$  of  $f$  and any positive integer  $n$ , there exists an open subset  $\mathcal{V} \subset \mathcal{U}$  such that every  $g \in \mathcal{V}$  satisfies property  $S_{n,p(g)}$ .*

The idea to prove this Proposition is to produce many nice horseshoes, as done by Downarowicz and Newhouse in [DN]. However, in their context, there is an abundance of homoclinic tangencies to produce such horseshoes. In our context we will use Lemma 8, which is a key and technical lemma, to overcome the lack of such abundance in general.

*Proof.* We can suppose that every periodic orbit of  $Per_h^n(f)$  and the orbit of  $p$  has an analytic continuation on  $\mathcal{U}$ . Moreover, by semicontinuity arguments, there exists  $k$  such that for every  $g \in \mathcal{U}$  we have

$$\#\{q \in Per_h^n(g); \text{ homoclinically related with } p\} = k.$$

We denote the elements of this set for  $f$  by  $\{p_1, \dots, p_k\}$ .

Since  $H(p, f) = H(p_1, f)$  admits no  $i$ -dominated splitting, Gourmelon's result [G] implies that, after some perturbation, we can suppose that  $f$  exhibits a homoclinic tangency for  $p_1$ , i.e., there exists a non transversal intersection between  $W^s(O(p_1), f)$  and  $W^u(O(p_1), f)$ .

Now we state a technical lemma.

**Lemma 8.** *If  $n$  is large enough, there exist a diffeomorphism  $g \in \mathcal{U}$  and a small neighborhood  $\mathcal{V} \subset \mathcal{U}$  of  $g$  such that for every  $h \in \mathcal{V}$  the items of property  $S_{n,p(h)}$  holds for  $p_1(h)$ , and moreover  $g$  exhibits a homoclinic tangency for  $p_2(g)$ .*

We postpone the proof of this lemma and finish the proof of the Proposition.

We consider now  $g_1$  and the neighborhood  $\mathcal{V}_1$  of  $g_1$  given by Lemma 8. Now, since  $g_1$  exhibits a homoclinic tangency for  $p_2(g_1)$ , and  $p_3(g_1)$  is homoclinically related to  $p_2(g_1)$ , we can use Lemma 8 again to obtain a diffeomorphism  $g_2$  and a neighborhood  $\mathcal{V}_2 \subset \mathcal{V}_1$  of  $g_2$  such that for every diffeomorphism  $h \in \mathcal{V}_2$  the items of property  $S_{n,p(h)}$  holds for  $p_2(h)$ , and moreover  $g_2$  exhibits a homoclinic tangency for  $p_3(g)$ .

Now, we repeat the process finitely many times, to obtain a diffeomorphism  $g = g_k$  and a neighborhood  $\mathcal{V} = \mathcal{V}_k \subset \mathcal{V}_{k-1} \dots \subset \mathcal{V}_1 \subset \mathcal{U}$  of  $g$  such that the items of property  $S_{n,p(h)}$  holds for  $p_i(h)$  with  $i = 1, \dots, k$  for every  $h \in \mathcal{V}$ . And then, by choice of  $\mathcal{U}$ , every diffeomorphism  $h \in \mathcal{V}$  satisfy property  $S_{n,p(h)}$ .  $\square$

**3.1. Proof of Lemma 8.** First of all, we observe that many times in this proof we use expressions like “by some perturbation”, or “we can perturb  $f$ ”, to say we can take a diffeomorphism arbitrary close to  $f$ . Sometimes, in order to not complicate the notation we use the same letter to denote the new diffeomorphism. Also, when we say “by a local perturbation” we mean that we can perform a perturbation of  $f$  keeping the new diffeomorphism equal to  $f$  outside some small open set.

Let  $q$  be a point of homoclinic tangency of  $p_1$ , and  $V$  be a small neighborhood of  $O(p_1)$  such that  $f^{-1}(q)$  is not in  $V$ . Shrinking  $V$ , if necessary, we can suppose  $f^{\tau(p_1, f)} = Df^{\tau(p_1, f)}$  (in local coordinates on  $V$ ) after a perturbation (see Franks' lemma [F]). We remark that after this perturbation the homoclinic tangency could

disappear. Nevertheless, since  $f^{-1}(q)$  is not in  $V$ , using the continuity of compact parts of unstable and stable manifolds of  $p_1$ , by a local perturbation in some neighborhood of  $f^{-1}(q)$  we can recover the homoclinic tangency.

Up to take another point of the orbit of  $q$ , we can suppose that  $q \in V$  and  $f^{-1}(q) \notin V$ . So, we can take a neighborhood  $U$  of  $q$  such that  $f^{-1}(U) \cap V = \emptyset$ . We denote by  $D$  the connected component of  $W^u(p_1, f) \cap U$  that contains  $q$ .

Now, we look to  $U$  in some local coordinates with the splitting  $T_q D \oplus T_q D^\perp$ , and such that  $q = 0$  in these coordinates. Since  $D \subset W^u(p_1, f)$  we have that  $D$  is a graph of a  $C^1$  map  $r : T_q D \rightarrow T_q D^\perp$ , i.e.  $D = (x, r(x))$ . Moreover,  $Dr(q)$  is close to zero. Hence, the diffeomorphism  $\phi(x, y) = (x, y - r(x))$  is  $C^1$  close to identity in a small neighborhood of  $q$ . In particular, there exists a diffeomorphism  $h$ ,  $C^1$ -close to identity, such that  $h = \phi$  in some small neighborhood of  $q$ , and  $h = Id$  for points far away from  $q$ . Thus,  $f_1 := h \circ f$  is a  $C^1$  local perturbation of  $f$  such that  $T_q D \cap U \subset W^u(p_1, f_1)$ . Since  $f^{-1}(U) \cap V = \emptyset$ , we have that  $f_1 = f$  in  $V$ , as a consequence  $f_1|_V$  is still linear, and  $W_{loc}^s(p_1, f_1)$  remains unchanged in  $U$ . Since  $q$  is a non transversal homoclinic point we have that  $T_q D \cap E^s(p_1, f)$  is a non trivial subspace. Actually, we can assume that  $T_q D \cap E^s(p_1, f)$  is an one-dimensional subspace, after some local perturbation if necessary. Thus,  $f_1$  exhibits an interval of homoclinic tangencies containing  $q$ .

Let  $I$  be this interval of homoclinic tangencies. Replacing the local coordinates in  $U$ , if necessary, we can suppose that  $\{(x_1, 0, \dots, 0), -3a \leq x_1 \leq 3a\} \subset I$ , for some  $a > 0$  small enough.

Let  $N$  be a large positive integer. Taking  $I$  smaller, if necessary, we can construct a diffeomorphism  $\Theta : M \rightarrow M$ , such that  $\Theta = Id$  in  $B(0, 2a)^c$  and

$$\Theta(x, y) = \left( x_1, \dots, x_s, y_1 + A \cos \frac{\pi x_1 N}{2a}, y_2, \dots, y_u \right), \text{ for } (x, y) \in B(0, a) \subset U,$$

for  $A = \frac{2Ka\delta}{\pi N}$ , where  $K$  is a constant which depends only on the local coordinates over  $U$  and  $\delta > 0$  is so small as we want.

Hence, taking  $g = \Theta \circ f_1$ , we have that  $g$  is  $\delta - C^1$  close to  $f_1$  and moreover  $g = f_1$  in the complement of  $f_1^{-1}(B(q, 2a))$ . Note that  $g$  depends on  $N$  but to not complicate the notation we denote this diffeomorphism by  $g$ , independent of  $N$ .

**Remark 9.** The most important properties of this new diffeomorphism is that  $g$  has  $N$  transversal homoclinic points for  $p_1$  inside  $U$ , but  $g$  still has an interval of homoclinic tangency inside  $U$ , in fact there are two intervals of homoclinic tangency in  $U$ : one inside  $\{(x_1, 0, \dots, 0), -3a \leq x_1 \leq -2a\}$  and other inside  $\{(x_1, 0, \dots, 0), 2a \leq x_1 \leq 3a\}$ , in local coordinates.

To simplify notation we assume  $p_1$  is a fixed point, being similar the general case.

We remark that  $g|_V$  is still linear in local coordinates, since  $f$  is equal  $g$  in  $V$ . Let  $D_t = D^s \times D_t^u$  be a small rectangle, with  $D^s = W_{loc}^s(p_1, g) \cap U$ , and  $D_t^u$  a small disk in  $\{(0, \dots, 0, y_1, \dots, y_n), y_i \in \mathbb{R}^+ \text{ and } |y_i| < A/4\}$ , such that  $t$  is the smallest positive integer such that  $g^t(D_t)$  is a disk  $A/4 - C^1$  close to the connected component of  $W^u(p_1, g) \cap U$  containing the  $N$  transversal homoclinic points built before. We remark that  $t$  depends on  $N$ , and  $t \rightarrow \infty$  when  $N \rightarrow \infty$ .

Observe that  $A$  is small if  $N$  is large, and by choice of  $D_t$ , we have that  $g(D_t) \cap D_t$  has  $N$  disjoint connected components. Moreover, we have that the maximal

invariant set in  $D_t$  for  $g^t$

$$\tilde{\Lambda}(p_1, N) = \bigcap_{j \in \mathbb{Z}} g^{tj}(D_t)$$

is a hyperbolic set inside  $H(p_1, g)$ .

Let  $\Lambda(p_1, N) = \bigcup_{0 \leq j \leq t} g^j(\tilde{\Lambda}(p_1, N))$  be the hyperbolic periodic set of  $g$  induced by  $\tilde{\Lambda}(p_1, N)$ . Since  $g|_{\tilde{\Lambda}(p_1, N)}$  is conjugated with the full shift of  $N$  symbols, we have that  $h(g|_{\Lambda(p_1, N)}) = \frac{1}{t} \log N$ ,

We recall that  $g|_V$  is linear. So, if  $m$  is the largest positive integer such that  $g^j(x) \in V$  for  $0 \leq j \leq m$ , there exist constants  $K_1$  and  $K_2$  depending on the local coordinate on  $V$  such that

$$(1) \quad K_1 \|Dg(p_1)^m |E^u|^{-1} \leq d(x, W_{loc}^s(p_1, g)) \leq K_2 \|Dg(p_1)^{-m} |E^u|,$$

for  $x \in V$ . Analogously, if  $m$  is the largest positive integer such that  $g^{-j}(x) \in V$  for  $0 \leq j \leq m$ , then there exist constants  $K_3$  and  $K_4$  such that

$$(2) \quad K_3 \|Dg(p_1)^{-m} |E^s|^{-1} \leq d(x, W_{loc}^u(p_1, g)) \leq K_4 \|Dg(p_1)^m |E^s|.$$

Another consequence of  $g|_V$  be linear and the choice of  $t$  is the following result, which also appears in [CT].

**Lemma 10** (Lemma 4.2 of [CT]). *For  $A$  and  $t$  defined as before, there exists a positive integer  $K_5$ , which is independent of  $A$ , such that*

$$A < K_5 \max\{\|Dg(p_1)^{-t} |E^u|\|, \|Dg(p_1)^t |E^s|\|\}.$$

Let  $n$  be a large positive integer. Since  $A = \frac{2Ka\delta}{\pi N}$ , using Lemma 10 and recalling that  $N \rightarrow \infty$  implies  $t \rightarrow \infty$ , we can select a large positive integer  $N$ , such that

$$\frac{1}{t} \log N > \min \left\{ \frac{1}{t} \log \|Dg(p_1)^{-t} |E^u|^{-1}\|, \frac{1}{t} \log \|Dg(p_1)^t |E^s|^{-1}\| \right\} - \frac{1}{2n}.$$

But, when  $t$  goes to infinity the above minimum converges to  $\chi(p_1, g)$ , by definition. Therefore, there exists a large positive integer  $N_1$  such that

$$\frac{1}{t} \log N_1 > \chi(p_1, g) - \frac{1}{n}.$$

So, it is possible to find a  $C^1$ -perturbation  $g$  of  $f$  such that

$$h(g|_{\Lambda(p_1, N_1)}) > \chi(p_1, g) - \frac{1}{n}.$$

Now, by the variational principle there exists an ergodic measure  $\mu_N \in \mathcal{M}(\Lambda(p_1, N))$  such that

$$(3) \quad h_{\mu_N}(g) > \chi(p_1, g) - \frac{1}{n}, \text{ for } N \geq N_1.$$

Observe that the orbit of points in the hyperbolic set  $\Lambda(g, N)$ , when  $N$  is large enough, stay almost all the time inside the neighborhood  $V$  of  $p_1$ , which one could be assumed so small as we wanted. Hence, there exists a positive integer  $N_2$  such that if  $\mu \in \mathcal{M}(f|_{\Lambda(g, N)})$  is ergodic then  $\rho(\mu, \mu_{p_1}) < 1/n$ , for every  $N \geq N_2$ .

Finally, we find  $N_3$  in order to obtain property (e) of  $S_{n,p(g)}$  for  $\Lambda(p_1, N)$  with  $N \geq N_3$ .



We define

$$V_k^u = V \cap g(V) \cap \dots \cap g^k(V), \text{ and}$$

$$V_k^s = V \cap g^{-1}(V) \cap \dots \cap g^{-k}(V).$$

Given vectors  $v, w \in \mathbb{R}^{2n}$  and subspaces  $E, F \subset \mathbb{R}^{2n}$  we define

$$\text{ang}(v, w) := \left| \tan \left[ \arccos \left( \frac{\langle v, w \rangle}{\|v\| \|w\|} \right) \right] \right|,$$

$$\text{ang}(v, E) = \min_{w \in E, \|w\|=1} \text{ang}(v, w) \quad \text{and} \quad \text{ang}(E, F) = \min_{w \in E, \|w\|=1} \text{ang}(w, F).$$

The following lemma, is also a straightforward consequence of  $g|V$  be linear, as in Lemma 4.4 in [CT].

**Lemma 11.** *With above definitions, there exists positive constants  $K_6$  and  $K_7$ , such that*

1) *if  $z \in V_k^u$ ,  $v \in \mathbb{R}^{2n} \setminus E_{p_1}^u$  and  $\text{ang}(g^{-k}(v), E_{p_1}^u) \geq 1$ , then*

$$K_6 \|Dg_{p_1}^k|E^s\|^{-1}|v| \min\{\text{ang}(v, E_{p_1}^u), 1\} \leq |Dg^{-k}(z)(v)| \leq K_7 \|Dg_{p_1}^{-k}|E^s\| \|v\|$$

2) *if  $z \in V_k^s$ ,  $v \in \mathbb{R}^{2n} \setminus E_{p_1}^s$  and  $\text{ang}(g^k(v), E_{p_1}^s) \geq 1$ , then*

$$K_6 \|Dg_{p_1}^{-k}|E^u\|^{-1}|v| \min\{\text{ang}(v, E_{p_1}^s), 1\} \leq |Dg^k(z)(v)| \leq K_7 \|Dg_{p_1}^k|E^u\| \|v\|$$

Now, since  $\Lambda(p_1, N) = \cup_{i=0}^{t-1} g^i(\tilde{\Lambda}(p_1, N))$  with  $\tilde{\Lambda}(p_1, N) \subset V$ , then we can take positive integers  $k$  and  $T$  such that  $t = k + T$ , and  $g^i(\tilde{\Lambda}(p_1, N)) \subset V$  for  $0 \leq i \leq k$ . Moreover, by construction of  $\tilde{\Lambda}(p_1, N)$  this  $T$  can be taken independent of  $N$ . Hence, provided  $t$  goes to infinity when  $N$  goes to infinity, we have that  $k$  also goes to infinity. Now, we know that the hyperbolic decomposition  $T_{\tilde{\Lambda}(p_1, N)} M = \tilde{E}^s \oplus \tilde{E}^u$  of the hyperbolic set  $\tilde{\Lambda}(p_1, N)$  is such that  $\tilde{E}^s(g^{-k}(z_1))$  and  $\tilde{E}^u(g^k(z_2))$  are close to  $E_{p_1}^s$  and  $E_{p_1}^u$ , respectively, for every  $z_1 \in g^k(\tilde{\Lambda}(p_1, N))$  and  $z_2 \in \tilde{\Lambda}(p_1, N)$ . In particular,

$$\text{ang}(Dg^{-k}(z_1)(v), E_{p_1}^u) > 1 \text{ for } v \in \tilde{E}^s(z_1) \text{ and}$$

$$\text{ang}(Dg^k(z_2)(v), E_{p_1}^s) > 1 \text{ for } v \in \tilde{E}^u(z_2).$$

Moreover, and the most important argument in this case, is that although  $\text{ang}(v, E_{p_1}^u)$  for  $v \in \tilde{E}^s(z_1)$ , and  $\text{ang}(v, E_{p_1}^s)$  for  $v \in \tilde{E}^u(z_2)$  are a very small constant, independent of  $N$ , we have ensured that

$$\text{ang}(Dg^{-k}(z_1)(v), E_{p_1}^u) > 1 \text{ and } \text{ang}(Dg^k(z_2)(v), E_{p_1}^s) > 1.$$

So, using these informations and Lemma 11 we can find constants  $K_6$  and  $K_7$ , such that for every  $z \in \tilde{\Lambda}(p_1, N)$ ,  $r = l(k + T)$  and for every  $l \in \mathbb{N}$ :

1) if  $v \in \tilde{E}^s(z)$  then

$$|Dg^{-r}(z)(v)| \geq (C_1 K_6)^l \|Dg_{p_1}^k|E^s\|^{-l}|v|$$

2) if  $v \in \tilde{E}^u(z)$  then

$$|Dg^r(z)(v)| \geq (C_1 K_7)^l \|Dg_{p_1}^{-k}|E^u\|^{-l}|v|,$$

where

$$C_1 = \inf_{z \in V \setminus g^{-1}(V), |v|=1} \|Dg^T(z)(v)\|.$$

Therefore, for  $N$  large enough, all points in  $\tilde{\Lambda}(p, N)$  have Lyapunov exponents with absolute values bigger than  $\chi(p, g) - 1/n$ . In particular, we can choose  $N_3$ , in order to get  $k \gg T$ , such that for any periodic point  $\tilde{q} \in \Lambda(p_1, N)$ , with  $N > N_3$ , we have

$$\chi(\tilde{q}) > \chi(p_1, g) - \frac{1}{n}.$$

Hence, if we take  $\Lambda(p_1, n) = \Lambda(p_1, N)$  for  $N = \max\{N_1, N_2, N_3\}$ , the items of property  $S_{n,p(g)}$  are satisfied for the perturbation  $g$  of  $f$  and the hyperbolic periodic point  $p_1$  of  $g$ .

**Remark 12.** By the construction of  $\Lambda(p_1, n)$ , observe that every item in the property  $S_{n,p}$  is robust. That is, if  $\tilde{g}$  is close to  $g$ , then the continuation of  $\Lambda(p_1, n)$  for  $\tilde{g}$  is such that all the items of property  $S_{n,p(\tilde{g})}$  is still true for  $p_1(\tilde{g})$ . This is because item (a) is a robust property; item (b) is a consequence of the continuation of  $\Lambda(p_1, n)$  be also inside  $V$  and item (c) still is true for continuations of  $\Lambda(p_1, n)$  by Lemma 11.

Hence, by the previous remark there exists a neighborhood  $\mathcal{V} \subset \mathcal{U}$  of  $g$ , such that every diffeomorphism  $h \in \mathcal{V}$  satisfies the items of property  $S_{n,p(h)}$  for  $p_1(h)$ .

Now, since the diffeomorphism  $g$  belongs to  $\mathcal{U}$ , we know that the hyperbolic periodic point  $p_2(g)$  still is homoclinic related with  $p_1(g)$ . Also, by Remark 9,  $g$  still exhibits a homoclinic tangency for  $p_1(g)$ . Now, by a perturbation using Franks Lemma, we can find a transversal homoclinic point to  $p_1(g)$ , such that the angle between  $W^s(p_1(g), g)$  and  $W^u(p_1(g), g)$  is so small as we want. Hence, since  $p_1(g)$  and  $p_2(g)$  are related, there exists a transversal homoclinic point for  $p_2(g)$  such that the angle between  $W^s(p_2(g), g)$  and  $W^u(p_2(g), g)$  is so small, too. Finally, using Franks Lemma once more, we can perturb  $g$  such that this transversal homoclinic point become a homoclinic tangency. Since this perturbation can be find in  $\mathcal{V}$ , we finish the proof.

#### 4. A LOCAL VERSION OF THE MAIN THEOREM

First we recall some knowns residual subsets. We denote by  $\mathcal{R}_1 \subset \text{Diff}^1(M)$  the residual subset given by [CMP], such that for every diffeomorphism  $g \in \mathcal{R}_1$  two homoclinic classes are either disjoint or coincide. By  $\mathcal{R}_2 \subset \text{Diff}^1(M)$  the residual subset given by [ABCDW], such that for every diffeomorphism  $g \in \mathcal{R}_2$ , every homoclinic class having a hyperbolic periodic point with index  $i$  and a hyperbolic periodic point with index  $j$ , with  $i < j$ , has a dense set of hyperbolic periodic points with index  $k$  for every  $i \leq k \leq j$ . And by  $KS$  the residual subset of Kupka-Smale diffeomorphisms. Hence, we define  $\mathcal{R}_4 = \mathcal{R}_1 \cap \mathcal{R}_2 \cap KS$ .

**Proposition 13.** *Let  $f \in \mathcal{R}_3$ ,  $p$  be a hyperbolic periodic point of  $f$ . If  $\mathcal{U}(f) \subset \text{Diff}^1(M)$  is a small enough neighborhood of  $f$ , there is a residual subset  $\mathcal{R} \subset \mathcal{U}(f)$  such that every  $g \in \mathcal{R}$  satisfies only one of the following statements:*

- (i)  $H(p_g, g)$  has a good decomposition;
- (ii)  $g|_{H(p_g, g)}$  has no symbolic extensions.

**Proof:**

Since  $f \in \mathcal{R}_3$  if  $\mathcal{U}(f)$  is small enough then there exist  $i$  and  $j$ , and hyperbolic periodic points  $p_i, p_{i+1}, \dots, p_j$  of  $f$  with  $\text{ind } p_k = k$ , for  $i \leq k \leq j$ , such that:

- $H(p, f) = H(p_i, f) = H(p_{i+1}, f) = \dots = H(p_j, f)$ ,
- for every hyperbolic periodic point  $q \in H(p, f)$  we have  $i \leq \text{ind } q \leq j$ .

By [ABCDW, Lemma 4.2, pg.20], there exists an open and dense subset of  $\mathcal{U}(f)$  over  $\mathcal{R}_3$  such that for every  $g$  in this subset, we still have that:

- $H(p(g), g) = H(p_i(g), g) = H(p_{i+1}(g), g) = \dots = H(p_j(g), g)$ ,
- for every hyperbolic periodic point  $q \in H(p(g), g)$  we have  $i \leq \text{ind } q \leq j$ .

Now, for any positive integer  $n$  and any  $i \leq k \leq j$ , we define  $\mathcal{B}_{n,p_k} \subset \mathcal{U}(f)$  as the subset of diffeomorphisms that robustly satisfies property  $S_{n,p_k}$ , i.e.,  $g \in \mathcal{B}_{n,p_k}$  if there is a small neighborhood of  $g$  where every diffeomorphism  $h$  satisfy property  $S_{n,p_k(h)}$ .

**Lemma 14.** *There is a residual subset of  $\mathcal{R}_4 \subset \mathcal{U}(f)$ , such that for any positive integer  $n$  and any  $i \leq k \leq j$ , if  $g \in \mathcal{R}_4$  and there is a sequence of diffeomorphisms  $\{g_m\} \in \mathcal{B}_{n,p_k}$  that converges to  $g$ , then  $g$  satisfies property  $S_{n,p_k(g)}$ .*

*Proof.* Let us define  $\mathcal{V}_{n,k} = \mathcal{B}_{n,p_k} \cup \overline{\mathcal{B}_{n,p_k}}^c$  be an open and dense subset in  $\mathcal{U}(f)$ , for every positive integer  $n$  and every  $i \leq k \leq j$ . Then,  $\mathcal{R}_4 = \bigcap_{n \geq 0} \bigcap_{i \leq k \leq j} \mathcal{V}_{n,k}$  is a residual subset in  $\mathcal{U}(f)$ . To finish the proof, let  $g \in \mathcal{R}_4$ . Given a positive integer  $n$  and  $i \leq k \leq j$ , if there exists diffeomorphisms  $g_m \in \mathcal{B}_{n,p_k}$  converging to  $g$ , then  $g \notin \overline{\mathcal{B}_{n,p_k}}^c$ . Therefore, since  $g \in \mathcal{V}_{n,k}$  we have that  $g \in \mathcal{B}_{n,p_k}$  and then satisfies property  $S_{n,p_k(g)}$ .  $\square$

Using Lemma 14, we define  $\mathcal{R} = \mathcal{R}_3 \cap \mathcal{R}_4$ , which is a residual subset in  $\mathcal{U}(f)$ . Now, we will verify that a diffeomorphism in this residual subset satisfies one of the two properties claimed in the proposition which finishes the proof.

For this we will use the following result of Burguet.

**Proposition 15** (Corollary 1 of [Bur1]). *Let  $f : M \rightarrow M$  be a dynamical system admitting a symbolic extension. Then the entropy function  $h : \mathcal{M}(f) \rightarrow \mathbb{R}$  is a difference of nonnegative upper semicontinuous functions. In particular the entropy function  $h$  restrict to any compact set of measures has a large set of continuity points.*

Let  $g \in \mathcal{R}$ , by choice of  $\mathcal{R}$ ,  $i$  and  $j$  are the two extreme indices in  $H(p(g), g)$  and  $H(p(g), g) = H(p_i(g), g) = H(p_{i+1}(g), g) = \dots = H(p_j(g), g)$ .

Suppose  $H(p(g), g)$  admits no good decomposition. Hence, there is some  $i \leq k \leq j$  such that  $H(p(g), g) = H(p_k(g), g)$  admits no  $k$ -dominated splitting.

By Proposition 7, for every  $n > 0$ , we can find a sequence of diffeomorphisms  $\{g_{n,m}\}_{m \in \mathbb{N}}$  converging to  $g$ , such that each  $g_{n,m} \in \mathcal{B}_{n,p_k}$ . Therefore, by Lemma 14,  $g$  satisfies property  $S_{n,p_k(g)}$  for every  $n > 0$ , since  $g \in \mathcal{R}_4$ .

We define  $\rho_0 = \max\{\chi(\tilde{p}, g); \tilde{p} \in \text{Per}_h(g) \text{ and related to } p_k(g)\}$ , and

$$\xi_1(g) = \left\{ \mu_{\tilde{p}} : \tilde{p} \in \text{Per}_h(g), \text{ related to } p_k(g) \text{ and } \chi(\tilde{p}, g) > \frac{\rho_0}{2} \right\}$$

which is a non empty subset in  $\mathcal{M}(f)$ . Then, we consider the compact subset  $\xi(g) = \overline{\xi_1(g)}$  in  $\mathcal{M}(g)$ .

Now, let  $\mu_{\tilde{p}} \in \xi_1$  and  $t$  be a positive integer. Since  $g$  satisfies property  $S_{n,p_k(g)}$  for every positive integer  $n$ , there exist ergodic measures  $\nu_m \rightarrow \mu_{\tilde{p}}$  such that  $h_{\nu_m}(g) >$

$\rho_0/2$ , for every  $m$ . Moreover, since these measures are supported on hyperbolic sets with the same index that  $p_k(g)$ , by Sigmund [SI], they are approximated by hyperbolic periodic measures also supported in these hyperbolic sets, and by item (c) of property  $S_{n,p_k(g)}$ , they belong to  $\xi_1(g)$ . Hence,  $\nu_m \in \xi(g)$  for every  $m$ , and then

$$\limsup_{\nu_m \rightarrow \mu_{\tilde{p}}, \nu_m \in \xi(g)} h_{\nu_m}(g) > \frac{\rho_0}{2}.$$

Therefore, since  $p$  is arbitrary and  $\xi_1(g)$  has dense periodic measures, there is no continuity point for the entropy function  $h$ . Thus, by Proposition 15, this implies that  $f$  has no symbolic extensions.

□

## 5. NON EXISTENCE OF SYMBOLIC EXTENSIONS VERSUS GOOD DECOMPOSITION

In this section we use Proposition 13 and the generic machinery to prove Theorem 2.

### Proof of Theorem 2:

Since  $\text{Diff}^1(M)$  is separable, there is a countable and dense subset  $\mathcal{A} \subset \text{Diff}^1(M)$ . Moreover, we can assume that  $\mathcal{A} \subset \mathcal{R}_3$ , the residual subset of  $\text{Diff}^1(M)$  in the hypothesis of Proposition 13.

Now, for any  $f \in \mathcal{A}$  and a small enough neighborhood  $\mathcal{U}(f)$  of  $f$ , we consider the residual subset  $\tilde{\mathcal{R}}_f$  in  $\mathcal{U}(f)$  given by Proposition 13. Thus, we define

$$\mathcal{R}_f = \tilde{\mathcal{R}}_f \cup (\mathcal{U}(f))^c,$$

which is a residual subset in  $\text{Diff}^1(M)$ , indeed. Also, since  $\mathcal{A}$  is a dense subset

$$\mathcal{U} = \bigcup_{f \in \mathcal{A}} \mathcal{U}(f),$$

is an open and dense subset of  $\text{Diff}^1(M)$ .

Finally, we define the following residual subset

$$\mathcal{R} = \bigcap_{f \in \mathcal{A}} \mathcal{R}_f \cap \mathcal{U}.$$

Now, let  $g \in \mathcal{R}$  and  $H(p, g)$  be a homoclinic class of  $g$ . Since  $g \in \mathcal{U}$ , there exists  $f \in \mathcal{A}$  such that  $g \in \mathcal{U}(f)$ , and then provided  $g$  also belongs to  $\mathcal{R}_f$ ,  $g$  should belong to  $\tilde{\mathcal{R}}_f$ . Therefore, by Proposition 13 we have that either  $H(p, g)$  has a good decomposition, or  $f|H(p, g)$  has no symbolic extensions. This completes the proof.

## 6. THE ISOLATED CASE

It is enough to prove that for  $f \in \mathcal{R}$  of Theorem 2, if  $H(p, f)$  has a good decomposition and it is isolated then it is partially hyperbolic. Let  $U$  be a neighborhood of  $H(p, f)$  such that  $H(p, f) = \bigcap_{n \in \mathbb{Z}} f^n(U)$ . Also, let  $E \oplus E_1 \oplus \cdots \oplus E_l \oplus F$  be the good decomposition. We will prove that  $E$  is contracting, a similar argument will prove that  $F$  is expanding.

We recall that  $\dim(E)$  is the smallest index of a periodic point in the class. By [ABCDW], there is another residual subset where we know that there exists a neighborhood  $\mathcal{U}$  of  $f$  such that  $\dim(E)$  is still the smallest index of a periodic point in the class  $H(p_g, g)$ , for any  $g \in \mathcal{U}$ , where  $p_g$  is the analytic continuation of  $p$ . The

intersection of this two residual subsets is the one claimed in the statement of the theorem.

Now, if  $E$  does not contract, using the Ergodic Closing Lemma, as Mañé did in [M2], then it is possible to find  $g \in \mathcal{U}$  with a periodic orbit  $O(q) \subset U$  with index smaller than  $\dim(E)$ .

However, we can consider the result of Abdenur proved in [Ab] to relative homoclinic class. Here, a relative homoclinic class of  $p$  for  $U$  is the subset of  $H(p, f)$  of points that have the whole orbit inside  $U$ , which we denote by  $H_U(p, f)$ . Therefore, since  $H_U(p, f) = H(p, f)$  isolated homoclinic class, for any  $h \in \mathcal{R}$  close enough to  $f$ ,  $H_U(p(h), h) = \bigcap_{n \in \mathbb{Z}} h^n(U)$  and then  $q \in H_U(p(h), h) \subset H(p(h), h)$ , which is a contradiction.

## 7. PROOF OF PROPOSITION 5

First, note that one inclusion is a directly consequence of the result of Liao, Viana and Yang [LVY]. More precisely, they have proved that far away from homoclinic tangency every diffeomorphism is  $h$ -expansive. For the other inclusion we will use Lemma 8.

Let  $f \in HT$ , that is,  $f$  exhibits a homoclinic tangency, say  $q$ , for a hyperbolic periodic point  $p$ . Given  $\epsilon > 0$  small, let us consider a small neighborhood  $\mathcal{U}$  of  $f$ , with  $\text{diam}(\mathcal{U}) < \epsilon$ . Then by Lemma 8 there is a perturbation  $f_1 \in \mathcal{U}$  of  $f$  such that  $f_1$  has a periodic hyperbolic basic set  $\Lambda_1$  satisfying

$$h(f_1|\Lambda_1) > \chi(p(f_1), f_1) - \frac{1}{n_0 + 1},$$

for a big positive integer  $n_0$  fixed, and moreover  $f_1$  still exhibits a homoclinic tangency for  $p(f_1)$ . As we can see in the proof of the Lemma 8,  $\Lambda_1$  can be found such that the base set  $\bar{\Lambda}_1$ , i.e.,  $\Lambda_1 = \bigcup f_1^j(\bar{\Lambda}_1)$ , is contained in a ball of radius so small, in particular, we can assume it is in a ball with radius  $\frac{1}{n_0+1}$ .

In the sequence, we consider a small neighborhood  $\mathcal{U}_1$  of  $f_1$ , such that for all diffeomorphisms in  $\mathcal{U}_1$  there is a continuation for  $\Lambda_1$ , and moreover  $\text{diam}(\mathcal{U}_1) < \frac{\epsilon}{n_0+1}$ . Now, using again Lemma 8 we can find a diffeomorphism  $f_2 \in \mathcal{U}_1$ , and a periodic hyperbolic basic set  $\Lambda_2$ , with base set contained in a ball with radius  $\frac{1}{n_0+2}$ , such that

$$h(f_2|\Lambda_2) > \chi(p(f_2), f_2) - \frac{1}{n_0 + 2},$$

and  $f_2$  still exhibits a homoclinic tangency for  $p(f_2)$ .

Following this process inductively we can find a sequence of diffeomorphism  $f_n \in \mathcal{U}_{n-1}$ , with  $\text{diam}(\mathcal{U}_n) < \frac{\epsilon}{n_0+n}$ ,  $\mathcal{U} \supset \mathcal{U}_1 \supset \dots \supset \mathcal{U}_n \supset \dots$ , and moreover, by construction,  $f_n$  is such that there exists periodic hyperbolic sets  $\Lambda_1, \dots, \Lambda_n$  with  $\text{diam}(\Lambda_i) < \frac{1}{n_0+i}$ , for every  $1 \leq i \leq n$ , and

$$h(f_n|\Lambda_i) > \chi(p(f_n), f_n) - \frac{1}{n_0 + i}.$$

Since this sequence of diffeomorphism is a Cauchy sequence, it converges to a diffeomorphism  $g$ , that is  $\epsilon$ -close to  $f$ . Now, by choice of the open sets  $\mathcal{U}_n$ ,  $g$  has periodic hyperbolic basic sets with diameter so small as we want with topological entropy away from zero, since  $\chi(p(f), f)$  varies continuously with the diffeomorphism  $f$ . Therefore,  $g$  can not be asymptotically  $h$ -expansive. And then, we have proved that  $HT \subset \overline{NAHE}$ .

## REFERENCES

- [A] M. Asaoka, Hyperbolic Sets Exhibiting  $C^1$ –Persistent Homoclinic Tangency for Higher Dimensions, Proceedings of the American Mathematical Society, Volume 136, n 2, Feb. 2008, Pages 677-686.
- [Ab] F. Abdenur, Generic Robustness of spectral decompositions, Ann. Scient. Éc. Norm. Sup., t.36, 2003, p. 213 à 224.
- [Ar] Arbieto, Alexander Periodic orbits and expansiveness. Math. Z. 269 (2011), no. 3-4, 801-807.
- [ABCDW] F. Abdenur, Ch. Bonatti, S. Crovisier, L. J. Díaz, L. Wen, Periodic points and homoclinic classes. Ergodic Theory Dynam. Systems 27 (2007), no. 1, 17-22.
- [BDP] Bonatti, C.; Daz, L. J.; Pujals, E. R. A  $C^1$ -generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. Ann. of Math. (2) 158 (2003), no. 2, 355-418.
- [BD] M. Boyle, T. Downarowicz, The entropy theory of symbolic extensions. Invent. Math. 156, 119–161 (2004)
- [BFF] M.Boyle, D.Fiebig and U.Fiebig, Residual entropy, conditional entropy and subshift covers, Forum Math. 14 (2002), 713-757.
- [Buz] J.Buzzi, Intrinsic ergodicity for smooth interval maps, Israel J. Math., 100 (1997), 125-161.
- [Bur1] D. Burguet, Symbolic extensions and continuity properties of the entropy, Archiv der Mathematik, 96 (2011), p.387-400.
- [Bur2] D. Burguet,  $C^2$  surface diffeomorphisms have symbolic extensions, Inventiones Mathematicae 186 (2011), p.191-236.
- [BF] D. Burguet and T. Fisher, Symbolic extensions for partially hyperbolic dynamical systems with 2-dimensional center bundle. *preprint* (2011).
- [C] T. Catalan, A generic condition for existence of symbolic extension of volume preserving diffeomorphisms, arXiv:1109.3080.
- [CT] T. Catalan, A. Tahzibi, A lower bound for topological entropy of generic non Anosov symplectic diffeomorphisms, arXiv:1011.2441.
- [CMP] C. M. Carballo, C. A. Morales, M. J. Pacifico, Homoclinic classes for generic  $C^1$  vector fields. Ergodic Theory Dynam. Systems 23 (2003), no. 2, 403-415.
- [CSY] S. Crovisier, M. Sambarino, D. Yang, Partial Hyperbolicity and Homoclinic Tangencies, arXiv:1103.0869.
- [DF] L. Diaz, T. Fisher, Symbolic Extensions and Partially Hyperbolic, Discrete and Continuous Dynamical Systems Volume 29, n 4, April 2011 pp. 1419-1441.
- [DFPV] L. Diaz, T. Fisher, M. Pacifico, and J. Vieitez, Entropy-expansiveness for partially hyperbolic diffeomorphisms, *preprint* (2010) arXiv:1010.0721.
- [D] T. Downarowicz, Entropy of a symbolic extension of a dynamical system. Ergodic Theory Dynam. Systems 21 (2001), no. 4, 1051-1070.
- [DN] T. Downarowicz, S. Newhouse, Symbolic extensions and smooth dynamical systems. *Invent. math.* 160, 453-499 (2005).
- [DM] Downarowicz, Tomasz; Maass, Alejandro Smooth interval maps have symbolic extensions: the antarctic theorem. Invent. Math. 176 (2009), no. 3, 617-636.
- [F] Franks, J. Necessary conditions for stability of diffeomorphisms. Trans. A.M.S. 158 (1971), 301-308.
- [G] N. Gourmelon, Generation of Homoclinic tangencies by  $C^1$ –perturbations. *Discrete and continuous Dynamical Systems* 26, 1-42 (2010).
- [GY] Yang, Dawei; Gan, Shaobo. Expansive homoclinic classes. Nonlinearity 22 (2009), no. 4, 729-733.
- [L] Li, Xiaolong; On R-robustly entropy-expansive diffeomorphisms. Bull. Braz. Math. Soc. (N.S.) 43 (2012), no. 1, 73-98.
- [LVY] Liao Gang, Marcelo Viana, Jiagang Yang, The Entropy Conjecture for Diffeomorphisms away from Tangencies, arXiv:1012.0514.
- [M1] Mañé, Ricardo Expansive diffeomorphisms. Dynamical systems?Warwick 1974, pp. 162-174. Lecture Notes in Math., Vol. 468, Springer, Berlin, 1975.
- [M2] Mañé M. An Ergodic Closing Lemma. The Annals of Mathematics 2nd Ser., Vol 116, No. 3. (Nov., 1982), 503-540.
- [PPV] Pacifico, M. J.; Pujals, E. R.; Vieitez, J. L. Robustly expansive homoclinic classes. Ergodic Theory Dynam. Systems 25 (2005), no. 1, 271-300.

- [PPSV] Pacifico, M. J.; Pujals, E. R.; Sambarino, M.; Vieitez, J. L. Robustly expansive codimension-one homoclinic classes are hyperbolic. *Ergodic Theory Dynam. Systems* 29 (2009), no. 1, 179-200.
- [PV] M. J. Pacifico, J. Vieitez, Robust entropy expansiveness implies generic domination. *Nonlinearity* 23 1971-1990 (2010).
- [SV1] Sambarino, Martín; Vieitez, José L. On  $C^1$ -persistently expansive homoclinic classes. *Discrete Contin. Dyn. Syst.* 14 (2006), no. 3, 465-481.
- [SV2] Sambarino, Martín; Vieitez, José L. Robustly expansive homoclinic classes are generically hyperbolic. *Discrete Contin. Dyn. Syst.* 24 (2009), no. 4, 1325-1333.
- [SI] K. Sigmund, Generic Properties of Invariant Measures for Axiom A-Diffeomorphisms, *Invent. Math.* 11 (1970), 99-109.
- [W] Wen, Lan Homoclinic tangencies and dominated splittings. *Nonlinearity* 15 (2002), no. 5, 1445-1469.

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, P. O. BOX 68530, 21945-970 RIO DE JANEIRO, BRAZIL.

*E-mail address:* `arbieto@im.ufrj.br`

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, P. O. BOX 68530, 21945-970 RIO DE JANEIRO, BRAZIL.

*E-mail address:* `almaar@im.ufrj.br`

FACULDADE DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE UBERLÂNDIA.

*E-mail address:* `tcatalan@famat.ufu.br`

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, P. O. BOX 68530, 21945-970 RIO DE JANEIRO, BRAZIL.

*E-mail address:* `senos@im.ufrj.br`